NONDEGENERACY OF HALF-HARMONIC MAPS FROM $\mathbb{R}$ INTO $\mathbb{S}^1$

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Abstract

We prove that the standard half-harmonic map $U : \mathbb{R} \to \mathbb{S}^1$ defined by

$$x \to \begin{pmatrix} \frac{x^2-1}{x^2+1} \\ \frac{-2x}{x^2+1} \end{pmatrix}$$

is nondegenerate in the sense that all bounded solutions of the linearized half-harmonic map equation are linear combinations of three functions corresponding to rigid motions (dilation, translation and rotation) of U.

1. Introduction

Due to their importance in geometry and physics, the analysis of critical points of conformal invariant Lagrangians has attracted much attention since 1950s. A typical example is the Dirichlet energy which is defined on two-dimensional domains and its critical points are harmonic maps. This definition can be generalized to even-dimensional domains whose critical points are called polyharmonic maps. In recent years, people are very interested in the analog of Dirichlet energy in odd-dimensional case, for example, [2], [3], [4], [5], [13], [14] and the references therein. Among these works, a special case is the so-called half-harmonic maps from $\mathbb{R}$ into $\mathbb{S}^1$ which are defined as critical points of the line energy

$$\mathcal{L}(u) = \frac{1}{2}\int_{\mathbb{R}} |(-\Delta_{\mathbb{R}})^{\frac{1}{4}} u|^2 dx. \tag{1.1}$$

Note that the functional $\mathcal{L}$ is invariant under the trace of conformal maps keeping invariant the half-space $\mathbb{R}^2_+$: the Möbius group. Half-harmonic maps have close relations with harmonic maps with partially free boundary and minimal surfaces with free boundary, see [12] and [13]. Computing the associated Euler-Lagrange equation of (1.1), we obtain that if $u : \mathbb{R} \to \mathbb{S}^1$ is a half-harmonic map, then u satisfies the following equation,

$$(-\Delta_{\mathbb{R}})^{\frac{1}{2}}u(x) = \left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|u(x)-u(y)|^2}{|x-y|^2}dy\right)u(x) \text{ in } \mathbb{R}. \tag{1.2}$$

It was proved in [13] that

Proposition 1.1. *([13]) Let $u \in \dot{H}^{1/2}(\mathbb{R},\mathbb{S}^1)$ be a non-constant entire half-harmonic map into $\mathbb{S}^1$ and u^e be its harmonic extension to $\mathbb{R}^2_+$. Then there exist $d \in \mathbb{N}$, $\vartheta \in \mathbb{R}$, $\{\lambda_k\}_{k=1}^d \subset (0,\infty)$ and $\{a_k\}_{k=1}^d \subset \mathbb{R}$ such that $u^e(z)$ or its complex conjugate equals to*

$$e^{i\vartheta}\prod_{k=1}^{d}\frac{\lambda_k(z-a_k)-i}{\lambda_k(z-a_k)+i}.$$

Furthermore,

$$\mathcal{E}(u,\mathbb{R}) = [u]^2_{H^{1/2}(\mathbb{R})} = \frac{1}{2}\int_{\mathbb{R}^2_+}|\nabla u^e|^2 dz = \pi d.$$

This proposition shows that the map $U : \mathbb{R} \to \mathbb{S}^1$

$$x \to \begin{pmatrix}\frac{x^2-1}{x^2+1}\\ \frac{-2x}{x^2+1}\end{pmatrix}$$

is a half-harmonic map corresponding to the case $\vartheta = 0$, $d = 1$, $\lambda_1 = 1$ and $a_1 = 0$. In this paper, we prove the nondegeneracy of U which is a crucial ingredient when analyzing the singularity formation of half-harmonic map flow. Note that U is invariant under translation, dilation and rotation, i.e., for $Q = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix} \in O(2)$, $q \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$, the function

$$QU\left(\frac{x-q}{\lambda}\right) = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}U\left(\frac{x-q}{\lambda}\right)$$

still satisfies (1.2). Differentiating with α, q and λ respectively and then set $\alpha = 0$, $q = 0$ and $\lambda = 1$, we obtain that the following three functions

$$Z_1(x) = \begin{pmatrix}\frac{2x}{x^2+1}\\ \frac{x^2-1}{x^2+1}\end{pmatrix}, \quad Z_2(x) = \begin{pmatrix}\frac{-4x}{(x^2+1)^2}\\ \frac{2(1-x^2)}{(x^2+1)^2}\end{pmatrix}, \quad Z_3(x) = \begin{pmatrix}\frac{-4x^2}{(x^2+1)^2}\\ \frac{2x(1-x^2)}{(x^2+1)^2}\end{pmatrix} \tag{1.3}$$

satisfy the linearized equation at the solution U of (1.2) defined as

$$\begin{aligned}(-\Delta_{\mathbb{R}})^{\frac{1}{2}}v(x) \;=\; & \left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right)v(x)\\ & + \left(\frac{1}{\pi}\int_{\mathbb{R}}\frac{(U(x)-U(y))\cdot(v(x)-v(y))}{|x-y|^2}dy\right)U(x) \quad \text{in } \mathbb{R}\end{aligned} \tag{1.4}$$

for $v : \mathbb{R} \to T_U\mathbb{S}^1$. Our main result is

Theorem 1.1. *The half-harmonic map* $U : \mathbb{R} \to \mathbb{S}^1$

$$x \to \begin{pmatrix} \frac{x^2-1}{x^2+1} \\ \frac{-2x}{x^2+1} \end{pmatrix}$$

is nondegenerate in the sense that all bounded solutions of equation (1.4) are linear combinations of Z_1*,* Z_2 *and* Z_3 *defined in (1.3).*

In the case of harmonic maps from two-dimensional domains into $\mathbb{S}^2$, the non-degeneracy of bubbles was proved in Lemma 3.1 of [7]. Integro-differential equations have attracted substantial research in recent years. The nondegeneracy of ground state solutions for the fractional nonlinear Schrödinger equations has been proved by Frank and Lenzmann [10], Frank, Lenzmann and Silvestre [11], Fall and Valdinoci [9], and the corresponding result in the case of fractional Yamabe problem was obtained by Dávila, del Pino and Sire in [6].

2. Proof of Theorem 1.1

The rest of this paper is devoted to the proof of Theorem 1.1. For convenience, we identify $\mathbb{S}^1$ with the complex unite circle. Since Z_1, Z_2 and Z_3 are linearly independent and belong to the space $L^\infty(\mathbb{R}) \cap Ker(\mathcal{L}_0)$, we only need to prove that the dimension of $L^\infty(\mathbb{R}) \cap Ker(\mathcal{L}_0)$ is 3. Here the operator $\mathcal{L}_0$ is defined as

$$\mathcal{L}_0(v) = (-\Delta_{\mathbb{R}})^{\frac{1}{2}} v(x) - \left(\frac{1}{2\pi} \int_{\mathbb{R}} \frac{|U(x) - U(y)|^2}{|x-y|^2} dy \right) v(x) - \left(\frac{1}{\pi} \int_{\mathbb{R}} \frac{(U(x) - U(y)) \cdot (v(x) - v(y))}{|x-y|^2} dy \right) U(x),$$

for $v : \mathbb{R} \to T_U\mathbb{S}^1$. Let us come back to equation (1.4), for $v : \mathbb{R} \to T_U\mathbb{S}^1$, $v(x) \cdot U(x) = 0$ holds pointwisely. Using this fact and the definition of $(-\Delta_{\mathbb{R}})^{\frac{1}{2}}$ (see [8]), we have

$$\begin{aligned}
(-\Delta_{\mathbb{R}})^{\frac{1}{2}} v(x) &= \left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right) v(x) \\
&\quad + \left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{(U(x)-U(y))\cdot(v(x)-v(y))}{|x-y|^2}dy\right) U(x) \\
&= \left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right) v(x) \\
&\quad + \left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{(U(x)-U(y))}{|x-y|^2}dy \cdot v(x)\right) U(x) \\
&\quad + \left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{(v(x)-v(y))}{|x-y|^2}dy \cdot U(x)\right) U(x) \\
&= \left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right) v(x) \\
&\quad + \left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{(v(x)-v(y))}{|x-y|^2}dy \cdot U(x)\right) U(x) \\
&= \left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right) v(x) \\
&\quad + \left((-\Delta_{\mathbb{R}})^{\frac{1}{2}} v(x)\cdot U(x)\right) U(x).
\end{aligned}$$

Therefore equation (1.4) becomes to

$$\begin{aligned}
(-\Delta_{\mathbb{R}})^{\frac{1}{2}} v(x) &= \left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{|U(x)-U(y)|^2}{|x-y|^2}dy\right) v(x) + \left((-\Delta_{\mathbb{R}})^{\frac{1}{2}} v(x)\cdot U(x)\right) U(x) \\
&= \frac{2}{x^2+1} v(x) + \left((-\Delta_{\mathbb{R}})^{\frac{1}{2}} v(x)\cdot U(x)\right) U(x). \qquad (2.1)
\end{aligned}$$

Next, we will lift equation (2.1) to $\mathbb{S}^1$ via the stereographic projection from $\mathbb{R}$ to $\mathbb{S}^1 \setminus \{pole\}$:

$$S(x) = \begin{pmatrix} \frac{2x}{x^2+1} \\ \frac{1-x^2}{x^2+1} \end{pmatrix}. \qquad (2.2)$$

It is well known that the Jacobian of the stereographic projection is

$$J(x) = \frac{2}{x^2+1}.$$

For a function $\varphi : \mathbb{R} \to \mathbb{R}$, define $\tilde{\varphi} : \mathbb{S}^1 \to \mathbb{R}$ by

$$\varphi(x) = J(x)\tilde{\varphi}(S(x)). \qquad (2.3)$$

Then we have

$$\begin{aligned}[(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{\varphi}](S(x)) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\tilde{\varphi}(S(x))-\tilde{\varphi}(S(y))}{|S(x)-S(y)|^2}dS(y) \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\frac{1+x^2}{2}\varphi(x)-\frac{1+y^2}{2}\varphi(y)}{\frac{4(x-y)^2}{(x^2+1)(y^2+1)}}\frac{2}{1+y^2}dy \\ &= \frac{1+x^2}{4\pi}\int_{\mathbb{R}} \frac{(1+x^2)\varphi(x)-(1+y^2)\varphi(y)}{(x-y)^2}dy \\ &= \frac{1+x^2}{2}(-\Delta_{\mathbb{R}})^{1/2}\left[\frac{x^2+1}{2}\varphi(x)\right] \\ &= \frac{1+x^2}{2}(-\Delta_{\mathbb{R}})^{1/2}\left[\tilde{\varphi}(S(x))\right].\end{aligned}$$

Therefore,

$$(-\Delta_{\mathbb{R}})^{1/2}\left[\tilde{\varphi}(S(x))\right] = J(x)[(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{\varphi}](S(x)).$$

Denote $v=(v_1,v_2)$ and let $\tilde{v}_1$, $\tilde{v}_2$ be the functions defined by (2.3) respectively. Then the linearized equation (2.1) becomes

$$\begin{cases} J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 = J(x)\tilde{v}_1 + \frac{x^2-1}{x^2+1}\frac{x^2-1}{x^2+1}J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \frac{x^2-1}{x^2+1}\frac{-2x}{x^2+1}J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2, \\ J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2 = J(x)\tilde{v}_2 + \frac{-2x}{x^2+1}\frac{x^2-1}{x^2+1}J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \frac{-2x}{x^2+1}\frac{-2x}{x^2+1}J(x)(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2. \end{cases}$$

Since $J(x)>0$ and set $U=(\cos\theta,\sin\theta)$, we get

$$\begin{cases} (-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 = \tilde{v}_1 + \cos^2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \cos\theta\sin\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2, \\ (-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2 = \tilde{v}_2 + \cos\theta\sin\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \sin^2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2, \end{cases}$$

which is equivalent to

$$\begin{cases} (-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 = 2\tilde{v}_1 + \cos 2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 + \sin 2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2, \\ (-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2 = 2\tilde{v}_2 + \sin 2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_1 - \cos 2\theta(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\tilde{v}_2. \end{cases}$$

Set $w=\tilde{v}_1+i\tilde{v}_2$, $z=\cos\theta+i\sin\theta$, then we have

$$(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}w = 2w + z^2(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}\bar{w}. \tag{2.4}$$

Here $\bar{w}$ is the conjugate of w.

Since $v\in L^\infty(\mathbb{R})$, w is also bounded, so we can expand w into fourier series

$$w=\sum_{k=-\infty}^{\infty} a_k z^k.$$

Note that all the eigenvalues for $(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}}$ are $\lambda_k = k$, $k = 0, 1, 2, \cdots$, see [1]. Using (2.4), $(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}} z^k = kz^k$ and $(-\Delta_{\mathbb{S}^1})^{\frac{1}{2}} \bar{z}^k = k\bar{z}^k$, we obtain

$$\begin{cases} (-k-2)a_k = (2-k)\bar{a}_{2-k}, \text{ if } k < 0, \\ (k-2)a_k = (2-k)\bar{a}_{2-k}, \text{ if } 0 \le k \le 2, \\ a_k = \bar{a}_{2-k}, \text{ if } k \ge 3. \end{cases}$$

Furthermore, from the orthogonal condition $v(x) \cdot U(x) = 0$ (so $(\tilde{v}_1, \tilde{v}_2) \cdot (\cos\theta, \sin\theta) = 0$), we have

$$a_k = -\bar{a}_{2-k}, \quad k = \cdots - 1, 0, 1, \cdots .$$

Thus

$$a_k = 0, \text{ if } k < 0 \text{ or } k \ge 3$$

and

$$a_0 = -\bar{a}_2, \quad a_1 = -\bar{a}_1$$

hold, which imply that

$$w = -\bar{a}_2 + a_1 z + a_2 z^2 = a(iz) + b\left[\frac{i}{2}(z-1)^2\right] + c\frac{(z^2-1)}{2}.$$

Here a, b, c are real numbers and satisfy relations

$$i(a-b) = a_1, \quad \frac{c}{2} + \frac{i}{2}b = a_2.$$

And it is easy to check that iz, $\frac{i}{2}(z-1)^2$ and $\frac{(z^2-1)}{2}$ are respectively Z_1, Z_2 and Z_3 under stereographic projection (2.2). By the one-to-one correspondence of w and v, we know that the dimension of $L^\infty(\mathbb{R}) \cap Ker(\mathcal{L}_0)$ is 3. This completes the proof.

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