LOCALLY CONFORMAL SYMPLECTIC BLOW-UPS

SONG YANG, XIANGDONG YANG, AND GUOSONG ZHAO

ABSTRACT. In this paper, we study the blow-up of a locally conformal symplectic manifold. We show that there exists a locally conformal symplectic structure on the blow-up of a locally conformal symplectic manifold along a compact induced symplectic submanifold.

1. INTRODUCTION

Let M be a smooth manifold. A symplectic form on M is a 2-form $\omega \in \Omega^2(M)$ satisfying: (1) $d\omega = 0$ and (2) ω is non-degenerate, i.e. the map

$$T_pM \ni v \longmapsto \omega(v, -) \in T_p^*M, \ \forall p \in M,$$

is an isomorphism. It is of importance to point out that the existence of the symplectic form ω on M determines pieces of topological data: the de Rham cohomology of M with even degrees are non-vanishing and the dimension of M is even, denoted by $2n$, and there exists a homotopy class of reductions of the structural group of the tangent bundle TM to $\mathrm{U}(n) \simeq \mathrm{Sp}(2n; \mathbb{R})$. In particular, if M is a complex manifold and ω is a Kähler form of a Hermitian metric on M then we say that (M, ω) a Kähler manifold.

In a more general setting, a subclass of almost symplectic manifolds called locally conformal symplectic manifolds (for short LCS) was introduced and studied by Lee [7], Liebermann[8] and Vaisman [13, 14]. Intuitively, a locally conformal symplectic form is a non-degenerate 2-form ω which is conformally equivalent to a symplectic form locally. From a conformal point of view, locally conformal symplectic manifolds may be thought of as closest to symplectic manifolds. In particular, the locally conformal symplectic manifolds can serve as natural phase spaces of Hamiltonian dynamical systems and from geometric aspect it appears in the study of contact manifolds and Jacobi manifolds

Date: August 19, 2016.

2010 *Mathematics Subject Classification.* 55B35, 55C55.

Key words and phrases. Locally conformal symplectic manifolds, Locally conformal symplectic blow-ups.

This work is partially supported by NSFC (grant No. 11571242) and the second author is also partially supported by FRFCU (project No. 0903005203329 and No. 106112016CDJXY100005).

(cf. [14, 1, 6]). Likewise, if M is a complex manifold and a locally conformal symplectic form ω on M is the Kähler form of a Hermitian metric h then we say that (M, ω) is a locally conformal Kähler manifold (for short LCK)(cf. [5]). To make this more precisely, we have the following diagram explaining the relationships between symplectic/Kähler manifolds and locally conformal symplectic/Kähler manifolds:

$$\begin{array}{ccc} \{\text{Kähler manifolds}\} & \subset & \{\text{LCK manifolds}\} \\ \cap & & \cap \\ \{\text{Symplectic manifolds}\} & \subset & \{\text{LCS manifolds}\}. \end{array}$$

It is well known that the blow-up operation is a very useful operation in symplectic/Kähler geometry. In particular, the Kähler property is preserved under blow-ups. In the symplectic category, it was McDuff [9] who first proved that the blow-up of a symplectic manifolds along a compact symplectic submanifold also admits a symplectic structure and using this symplectic blow-up technique she constructed the first simply-connected, symplectic manifolds which are non-Kähler. For locally conformal Kähler manifolds, Tricerri [12] and Vuletescu [15] proved that the blow-up of a locally conformal Kähler manifold at a point is locally conformal Kähler. In 2013, using the current theory on locally conformal Kähler manifolds, Ornea-Verbitsky-Vuletescu [11] showed that the blow-up of an locally conformal Kähler manifold along a submanifold is locally conformal Kähler if and only if the submanifold is globally conformally equivalent to a Kähler submanifold. In the locally conformal symplectic case, Y. Chen and the first named author [4] introduced the definition of locally conformal symplectic blow-up at points and proved that the locally conformal symplectic blow-ups at points also admits locally conformally symplectic structures. Therefore, a natural problem is: *What is the locally conformal symplectic blow-up along a submanifold?*

The purpose of this paper is to study some birational properties of locally conformal symplectic manifolds. Motivated by the work of McDuff [9] we give the construction of locally conformal symplectic blow-up. In addition, using the same method of McDuff [9] and Ornea-Verbitsky-Vuletescu [11] we prove the following result

Theorem 1.1. *Let (M, ω, θ) be a locally conformal symplectic manifold and Z be a compact induced globally conformally symplectic submanifold of M, and let $\pi : \tilde{M} \longrightarrow M$ be the blow-up of M along Z. Then $\tilde{M}$ also admits a locally conformally symplectic structure $(\tilde{\omega}, \tilde{\theta})$ where $\tilde{\theta} = \pi^*\theta$.*

This paper is organized as follows. We devote Section 2 to preliminaries of locally conformal symplectic structures. In Section 3, we give the construction of locally conformal symplectic blow-up. This construction is based on the fact that the tangent bundle of a locally conformal symplectic manifold is a symplectic vector bundle. In Section 4, we give the proof of the main result (Theorem 1.1). At last, we propose two further problems related to the locally conformal symplectic blow-ups.

2. Locally conformal symplectic manifolds

In this section we give a rapid review of locally conformal symplectic manifolds. Assume that M be a smooth manifold of dimension $n \geq 4$. Intuitively, a *locally conformally symplectic structure* on a manifold M is a non-degenerate 2-form ω which is locally conformal to a symplectic form. More precisely, if there exist an open covering $\{U_\alpha\}$ of M and a family of smooth functions a family of smooth real-valued functions $\{f_\alpha : U_\alpha \longrightarrow \mathbb{R}\}$ such that $\exp(-f_\alpha)(\omega|_{U_\alpha})$ is a symplectic form on U_α, i.e., $d(\exp(-f_\alpha)\omega|_{U_\alpha}) = 0$, then we say that ω is a locally conformal symplectic structure on M.

Let $\omega_\alpha := \exp(-f_\alpha)(\omega\,|_{U_\alpha})$, then from definition we have

$$\begin{aligned} 0 = d\omega_\alpha &= d(\exp(-f_\alpha)\omega) \\ &= -\exp(-f_\alpha)(df_\alpha \wedge \omega - d\omega) \\ &= \exp(-f_\alpha)(d\omega - df_\alpha \wedge \omega). \end{aligned}$$

on U_α. This implies that

$$d\omega = df_\alpha \wedge \omega \tag{2.1}$$

on U_α. Likewise, consider the form $\omega_\beta := \exp(-f_\beta)(\omega\,|_{U_\beta})$ we get

$$d\omega = df_\beta \wedge \omega \tag{2.2}$$

on U_β. Suppose that $U_\alpha \cap U_\beta \neq \emptyset$ then from (2.1) and (2.2) we obtain

$$(df_\alpha - df_\beta) \wedge \omega = 0 \tag{2.3}$$

on $U_\alpha \cap U_\beta$. Note that ω is non-degenerate and the wedge product with ω is injective on 1-forms, hence we obtain a globally defined closed 1-form $\theta := \{df_\alpha, U_\alpha\}$ on M which satisfies

$$d\omega = \theta \wedge \omega. \tag{2.4}$$

Equivalently, we have

Definition 2.1 (Locally conformal symplectic structure)**.** Let M be a smooth manifold of dimension $n \geq 4$. We say that a non-degenerate 2-form ω is a *locally conformally symplectic structure* (for short *LCS structure*) if, there exists a closed 1-form θ such that

$$d\omega = \theta \wedge \omega. \tag{2.5}$$

The triple (M, ω, θ) is called a locally conformally symplectic manifold.

Suppose that there exists another θ' satisfying (2.5), then $(\theta - \theta') \wedge \omega = 0$. From Cartan lemma we get $\omega = (\theta - \theta') \wedge \beta$ for some 1-form β; however, this leads to a contradiction with the non-degeneracy of ω. This implies that θ is uniquely determined by ω and we call it the *Lee form* of the LCS manifold. In particular, if θ is an exact 1-form, i.e. $\theta = df$ for some smooth function f on M then ω is called *globally conformally symplectic* (for short *GCS*) and it is straightforward to verify that $e^{-f}\omega$ is a symplectic form on M.

Example 2.2. Every locally conformal Kähler manifold is a locally conformal symplectic manifold. In particular, many well-known non-Kähler manifolds are locally conformal Kähler such as the Hopf manifolds and the Inoue surfaces and so on (cf. [5, Chapter 3]).

Example 2.3. Let N be a smooth manifold. Then the cotangent bundle (T^*N, ω) is an open symplectic manifold with the symplectic form $d\lambda$, where λ is the canonical 1-form on T^*N. If θ' is a closed 1-form on N, then $\omega := d_{\pi^*\theta'}\lambda$ is a locally conformally symplectic form on T^*N with the Lee form $\theta = \pi^*\theta'$, where $\pi : T^*N \longrightarrow N$ is the bundle map. Moreover, if θ' is an exact 1-form then $\omega = d_{\pi^*\theta'}\lambda$ is a globally conformally symplectic form.

Example 2.4. ([1, Section 5]) Let X be a compact contact manifold and let $\phi : X \longrightarrow X$ be a strict contactomorphism, there there exists a natural locally conformal symplectic structure on the mapping torus of X with respect to ϕ. In particular, we can choose a 3-dimensional contact X such that $X \times S^1$ admits no symplectic and complex structures. This gives rise to an example which is locally conformal symplectic and not locally conformal Kähler.

Let $\Omega^*(M)$ be the space of smooth forms on the locally conformal symplectic manifold (M, ω, θ). We may define the Lichnerowicz differential [1] by

$$\begin{aligned} d_\theta : \Omega^*(M) &\to \Omega^{*+1}(M) \\ \alpha &\mapsto d\alpha - \theta \wedge \alpha. \end{aligned}$$

Furthermore, we have a complex

$$\cdots \xrightarrow{d_\theta} \Omega^{k-1}(M) \xrightarrow{d_\theta} \Omega^k(M) \xrightarrow{d_\theta} \cdots \tag{2.6}$$

[1] In the case of locally conformal Kähler manifolds the differential is called the θ-twisted differential and the associated complex(cohomology) is called the Morse-Novikov complex (cohomology).

The complex $(\Omega^*(M), d_\theta)$ is called the *Lichnerowicz complex*, and the associated cohomology group

$$H^*_\theta(M) := H^*(\Omega^*(M); d_\theta)$$

is called the *Lichnerowicz cohomology*. This cohomology is a conformal invariant of the locally conformal symplectic manifold, which is a proper tool in the study of locally conformal symplectic geometry.

3. Construction of locally conformal symplectic blow-ups

In this section, inspired by McDuff's construction of symplectic blow-ups, we give the construction of blow-up of locally conformally symplectic manifolds along its induced locally conformal symplectic submanifolds and for more details, we refer to McDuff [9, Section 2 and Section 3].

Let (M, ω, θ) be a LCS manifold of dimension $2n$. Then for any $p \in M$ the tangent space T_pM is a symplectic vector space with the symplectic bilinear form

$$\omega_p : T_pM \times T_pM \longrightarrow \mathbb{R}.$$

This implies that the structural group of the tangent bundle of M is $\mathrm{Sp}(2n;\mathbb{R})$; in further, fix an orientation of M then we can reduce the structural group $\mathrm{Sp}(2n;\mathbb{R})$ to $U(n)$.

Definition 3.1 (Induced LCS submanifold)**.** Let (M, ω, θ) be a LCS manifold, and let $i : Z \hookrightarrow M$ be a submsnifold. We say that Z is an induced locally conformal symplectic submanifold (for short *ILCS* submanifold) if $i^*\omega$ is non-degenerate.

Definition 3.2 (Induced GCS submanifold)**.** We say that Z is an induced globally conformal symplectic submanifold (for short *IGCS* submanifold) if Z is a ILCS submanifold and the cohomology class $i^*[\theta] = 0$ vanishes.

Notice that a IGCS submanifold of a LCS manifold is always a symplectic submanifold. Now let (M, ω, θ) be a LCS manifold, and let $i : Z \hookrightarrow M$ be its ILCS submsnifold then we have the following lemma.

Lemma 3.3. *Let (M, ω, θ) be a LCS manifold, and let $Z \subset M$ be an ILCS submsnifold. Then the normal bundle $\mathcal{N} := \mathcal{N}_{Z/M}$ of Z in M admits a complex vector bundle structure.*

Proof. Note that the locally conformal symplectic form ω on M yields a smooth section of the vector bundle $T^*M \wedge T^*M$. The non-degeneration of ω means that (TM, ω) is a symplectic vector bundle. Since Z is an ILCS submsnifold of

M the tangent subbuncle $(TZ, \omega|_Z)$ is a symplectic subbundle of $(TM|_Z, \omega|_Z)$. Define the symplectic complement of TZ in $(TM|_Z, \omega|_Z)$ to be the space

$$TZ^\omega := \bigcup_{p\in Z}\{v \in T_pM \mid \omega_p(v, w) = 0, \forall w \in T_pZ\}.$$

On the one hand, we observe that TZ^ω is a symplectic vector bundle with symplectic bilinear form $\omega\mid_Z$ which can be identified with the normal bundle $\mathcal{N}$. On the other hand, since we can choose a compatible complex structure on each symplectic vector bundle to make it into a complex vector bundle. This immediately implies that the normal bundle $\mathcal{N}$ admits a complex vector bundle structure. □

We are now in a position to give the construction of LCS blow-up. This construction is analogous to the case of symplectic blow-up since the normal bundle $\mathcal{N}$ is a complex vector bundle. In the rest of this section we follow the lines in [9] and use the same results and intermediate steps as [9] to construct the LCS blow-ups.

Let $p : \mathbb{P}(\mathcal{N}) \longrightarrow Z,$ be the projective bundle corresponding to the normal bundle $\mathcal{N} \longrightarrow Z$. The tautological line bundle over $\mathbb{P}(\mathcal{N})$, denoted by L, is defined to be the subbundle of $\mathbb{P}(\mathcal{N}) \times \mathcal{N}$ whose fiber is $\{(l, v) \mid v \in l\}$, i.e.,

$$L := \{(l, v) \mid (l, v \in l) \in \mathbb{P}(\mathcal{N}) \times \mathcal{N}\}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc} L_0 & \longrightarrow & L & \xrightarrow{q} & \mathbb{P}(\mathcal{N}) \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & {\scriptstyle p}\downarrow \\ \mathcal{N}_0 & \longrightarrow & \mathcal{N} & \xrightarrow{\varphi} & Z \end{array}$$

where q and π are the projections of L over $\mathbb{P}(\mathcal{N})$ and $\mathcal{N}$ respectively, and L_0 is the complement of the zero section in L and $\mathcal{N}_0$ is the complement of the zero section in $\mathcal{N}$.

To define the blow-up as a smooth manifold, we need following notations:

- W a compact tubular neighborhood of Z in M.
- D a compact neighborhood of Z in $\mathcal{N}$ diffeomorphic to W
- $\tilde{D} := \pi^{-1}(D)$; it is a disc subbundle of the complex line bundle L.

Following McDuff [9] we have:

Definition 3.4 (LCS blow-up)**.** Let (M, ω, θ) be a LCS manifold, and let $Z \subset M$ be an ILCS submsnifold. The blow-up $\tilde{M}$ of M along Z is the manifold

$$\tilde{M} := \overline{M - W} \bigcup_{\partial\tilde{D}} \tilde{D},$$

where $\partial\tilde{D}$ is identified with ∂W via the diffeomorphism from D to W.

In particular, the map π gives rise to an identification of $\tilde{D} - \mathbb{P}(\mathcal{N})$ with $D - Z$, and thus an identification of $\tilde{M} - \mathbb{P}(\mathcal{N})$ with $M - Z$. Therefore, on topology we may view

$$\tilde{M} := (M - Z) \bigcup \tilde{D}$$

by equalizing $M - Z$ and $\tilde{D}$ along $W - Z \cong D - Z \cong \tilde{D} - \mathbb{P}(\mathcal{N})$. There is a natural inclusion $\mathbb{P}(\mathcal{N}) \hookrightarrow \tilde{M}$, and we call the projective bundle $\mathbb{P}(\mathcal{N})$ the *exceptional divisor* of the blow-up $\pi : \tilde{M} \longrightarrow M$ along Z.

Remark 3.5. Note that the construction of LCS blow-up depends on the complex vector bundle structure of the normal bundle $\mathcal{N}$ and the tubular neighbourhoods. Thus this construction is not canonical; however, we can choose the compact tubular neighborhood W of Z in M sufficiently small.

4. Proof of the main result

In this section we give the proof of Theorem 1.1. We use the same method as [9, Section 3] and for the reader's convenience we first recall this argument.

Let (U, ω) be a symplectic manifold and let $i : Z \hookrightarrow U$ be a compact symplectic submanifold of codimension $2k$. Consider the normal bundle $\pi : \mathcal{N} \longrightarrow Z$ of Z in U. Since $\mathcal{N}$ has a complex vector bundle structure the fiber $\mathcal{N}_x$ for each $x \in Z$ admits a canonical exact symplectic form. From another aspect, in the horizonal direction the zero section of $\mathcal{N}$, still write as Z, is a symplectic manifold with the symplectic form $\omega_Z := i^*\omega$. Choose a local trivialization of $\mathcal{N}$, i.e. an open covering $\{U_i\}$ of Z such that $\mathcal{N} |_{U_i} \cong U_i \times \mathbb{C}^k$. For each i there exists a 1-form α_i on $\mathcal{N} |_{U_i}$ satisfying:

(1) for any $x \in U_i$ the restriction of $d\alpha_i$ on the fiber $\mathcal{N}_x$ is the canonical form;
(2) α_i is zero on U_i.

Let $\{f_i\}$ be a partition of unity subordinate to the open cover $\{U_i\}$ then we may construct a closed 2-form on $\mathcal{N}$, denoted by

$$\rho = \pi^*\omega + \sum_i d(f_i \alpha_i).$$

In particular, ρ restrict to the canonical form on each fiber and to ω_Z on Z.

According to [9, Lemma 3.2], there exists a closed 2-form α on $\mathbb{P}(\mathcal{N})$ such that α restricts to the Kähler form of the canonical Fubini-Study metric on each fibre of $p : \mathbb{P}(\mathcal{N}) \longrightarrow Z$ and the pull-back of α under q^* is an exact form on L_0. Since $q^*\alpha$ is exact on L_0 we have $q^*\alpha = d\beta$ for some 1-form β on L_0. Let $\tilde{U} := \overline{U - W} \bigcup_{\partial\tilde{D}} \tilde{D}$ be the symplectic blow-up of U along Z. We can choose a constant $\varepsilon = \varepsilon(\rho, \alpha) > 0$ and a bump-function b on $\tilde{D}$ which equals 0 near $\partial\tilde{D}$. Define a closed 2-form $\tilde{\rho}$ on $\tilde{D}$ by setting

$$\tilde{\rho} := \begin{cases} \pi^*\rho + \varepsilon q^*\alpha & \text{on } \mathbb{P}(\mathcal{N}), \\ \pi^*\rho + \varepsilon d(b\beta) & \text{on } \tilde{D} - \mathbb{P}(\mathcal{N}). \end{cases}$$

We may choose suitable ε such that $\tilde{\rho}$ is non-degenerated on $\tilde{V} := \pi^{-1}(V)$, where V is a neighborhood of Z. Hence the 2-form

$$\tilde{\omega} := \begin{cases} \omega & \text{on } U - W \\ \tilde{\rho} & \text{on } \tilde{D} \end{cases}$$

is non-degenerated and closed, i.e. it is a symplectic form on $\tilde{U}$. More precisely, we have the following key result in the symplectic blow-up.

Proposition 4.1. *(*[9, Proposition 3.7]*) Suppose (U, ω) be a symplectic manifold, and $i : Z \hookrightarrow U$ be a compact symplectic submanifold (i.e., $i^*\omega$ is a symplectic form). Let $\pi : \tilde{U} \longrightarrow U$ be the blow-up of U along Z. Then there exists a symplectic form $\tilde{\omega}$ on $\tilde{U}$ such that*

$$\tilde{\omega}|_{\tilde{U}-\pi^{-1}(V)} = \pi^*\omega,$$

for some neighborhood V of Z.

We are in a position to prove Theorem 1.1. Assume that (M, ω, θ) is a locally conformal symplectic manifold. Let $Z \subset M$ be an induced globally confomal symplectic submanifold, thus the restriction of the Lee form $\theta|_Z$ is exact. By a conformal rescaling of the LCS form ω we may assume that $\theta|_Z = 0$. In fact, if $\theta|_Z = df$, we denote $\omega' := \exp(-f)\omega$, then $d\omega' = \exp(-f)(-df \wedge \omega + \theta \wedge \omega) = (\theta - df) \wedge \omega = 0$. Actually, an induced globally conformal symplectic submanifold is a symplectic submanifold. In the rest of this section we will prove the following

Theorem 4.2 (Theorem 1.1)**.** *Assume that (M, ω, θ) be a locally conformal symplectic manifold and $i : Z \hookrightarrow M$ is a compact induced symplectic submanifold. Let $\pi : \tilde{M} \longrightarrow M$ be the locally conformal symplectic blow-up of M along Z. Then $\tilde{M}$ also admits a locally conformal symplectic structure $\tilde{\omega}$ with the Lee form $\tilde{\theta} = \pi^*\theta$.*

Proof. By assumpation, the pull back $\theta|_Z := i^*\theta$ is zero. Let U be a neighborhood of Z such that the inclusion $i : Z \hookrightarrow U$ induces an isomorphism on the first de Rham cohomology groups

$$i^* : H^1_{dR}(U) \xrightarrow{\cong} H^1_{dR}(Z).$$

Via a conformal change of the LCS form ω, we may assume that $\theta|_U = 0$. It follows that $\omega|_U$ is a symplectic form on U. In particular, since $\theta|_U = 0$ the intersection of the support of θ with U is empty. Choose a sufficiently small tubular neighborhood of Z in M such that $W \subset U$. Let $\pi : \tilde{M} \longrightarrow M$ be the LCS blow-up of M along Z with respect to the compact tubular neighborhood W and a compact neighborhood D of Z in $\mathcal{N}$ which is diffeomorphic to W. Let $\tilde{U} := \pi^{-1}(U)$ then $\pi : \tilde{U} \longrightarrow U$ is the symplectic blow-up of U along Z with respect to the compact neighborhoods W and D, i.e.

$$\tilde{U} := \overline{U - W} \bigcup_{\partial \tilde{D}} \tilde{D}.$$

From Proposition 4.1, there exists a symplectic form $\tilde{\omega}_U$ on $\tilde{U}$, which is equal to $\pi^*\omega$ outside of $\pi^{-1}(V) \supset \pi^{-1}(Z) = \mathbb{P}(\mathcal{N})$ for a neighborhood V of Z in U. Observe that π gives rise to an identification between $\tilde{M} - \mathbb{P}(\mathcal{N})$ and $M - Z$; therefore, we obtain a non-degenerate 2-form $\tilde{\omega}$ on $\tilde{M}$ given by

$$\tilde{\omega} := \begin{cases} \pi^*\omega & \text{on } \tilde{M} - \tilde{U} \\ \tilde{\omega}_U & \text{on } \tilde{U}. \end{cases}$$

It remains to verify that $\tilde{\omega}$ is a LCS form with Lee form $\tilde{\theta} = \pi^*\theta$. It is straightforward since we have $\tilde{\theta} \mid_{\tilde{U}} = 0$ and $\tilde{\omega} = \pi^*\omega$ outside of $\tilde{U}$. This completes the proof. □

Under the locally conformal symplectic blow-ups we also have a blow-up formula of the Lichnerowicz cohomology as following.

Corollary 4.3. *(*[16, Theorem 1.1]*) Let (M, ω, θ) be a compact locally conformal symplectic manifold of dimension $2n$. Assume that $Z \subset M$ is a compact induced globally conformal symplectic submanifold of codimension $2r$. Then we have*

$$H^k_\theta(M) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}_{dR}(Z) \right) \cong H^k_{\tilde{\theta}}(\tilde{M}),$$

where $\pi : \tilde{M} \longrightarrow M$ is the locally conformal symplectic blow-up of M along Z.

5. Further problems

In [11, Corollary 2.11], using the current theory on complex manifolds, Ornea-Verbitsky-Vuletescu proved that if the blow-up of a compact locally conformal Kähler manifold along a compact submanifold admits a locally conformal Kähler structure then the submanifold must be an induced globally conformal Kähler submanifold. Similarly, for LCS manifolds we have the following problem:

If the blow-up of a compact LCS manifold along a compact ILCS submanifold admits a LCS structure, is it true that this submanifold is IGCS?

It is worthwhile to point out that for LCS manifolds we can not use the current theory since the almost complex structures on LCS manifolds are not integrable necessarily.

The existence of Kähler metrics on a manifold implies many topological properties, for instance, the even property of odd Betti numbers and formality property and so on. These properties enable people to construct many examples of non-Kähler and symplectic manifolds. In the case of locally conformal Kähler geometry, it is not easy to exclude whether a manifold admits a locally conformal Kähler metric for the lack of topological obstructions. Comparing with symplectic/Kähler geometries, Ornea-Verbitsky [10] proposed an open problem:

Construct a compact LCS manifold which admits no LCK metrics.

In 2011, Bande-Kotschick [1] constructed a 4-dimensional product manifold $M \times S^1$ which is LCS and not LCK. Later, in 2014 Bazzoni-Marrero [2] constructed a symplectic, and hence locally conformal symplectic, nilmanifold N which is not the product of a compact 3-manifold and a circle (see also Bazzoni-Marrero [3, Corollary 3.6]). In particular, they proved that N admits no complex structures. This implies that N is LCS and not LCK. In fact, using the LCS blow-up technique at points of LCS manifolds having no LCK structures, we may obtain more LCS manifolds without any LCK structures (cf. [4, Corollary 2.4]). Furthermore, a natural problem is:

How to construct examples of LCS manifolds which are not symplectic and LCK? Moreover, can we construct higher dimensional LCS manifolds without any complex structures?

References

[1] G. Bande, D. Kotschick, *Contact pairs and locally conformally symplectic structures*, Harmonic Maps and Differential Geometry, 85-98, Contemp. Math., 542, Amer. Math. Soc., Providence, RI, 2011.

[2] G. Bazzoni, J. C. Marrero, *Locally conformal symplectic nilmanifolds with no locally conformal Kähler metrics*, arXiv: 1407.5510v1.

[3] G. Bazzoni, J. C. Marrero, *On locally conformal symplectic manifolds of the first kind*, arxiv 1510.04947v2.

[4] Y. Chen, S. Yang, *On the blow-up of points on locally conformally symplectic manifolds*, C. R. Acad. Sci. Paris 354 (2016) 411-414.

[5] S. Dragomir, L. Ornea, *Locally conformal Kähler geometry*, Progress in Mathematics, Vol. 155, Birkhäuser, Boston, Basel, (1998).

[6] F. Guedira, A. Lichnerowicz, *Géomegrie des algèbres de Lie locales de Kirillov*, J. Math. Pures Appl. 63 (1984) 407-484.

[7] H. C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus*, Amer. J. Math. 65 (1943) 433-438.

[8] P. Libermann, *Sur le problème déquivalence de certaines structures infinitésimales régulières*, Ann. Mat. Pura. Appl. 36 (1954) 27-120.

[9] D. McDuff, *Examples of symplectic simply connected manifolds with no Kähler structure*, J. Differential Geom. 20 (1984) 267-277.

[10] L. Ornea, M. Verbitsky, *A report on locally conformally Kähler manifolds*, Harmonic Maps and Differential Geometry, 135-149, Contemp. Math., 542, Amer. Math. Soc., Providence, RI, 2011.

[11] L. Ornea, M. Verbitsky and V. Vuletescu, *Blow-ups of locally conformally Kähler manifolds*, Int. Math. Res. Not. 12 (2013) 2809-2821.

[12] F. Tricerri, *Some examples of locally conformal Kähler manifolds*, Rend. Sem. Mat. Univ. Politec. Torino 40 (1982) 81-92.

[13] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. 24 (1976) 338-351.

[14] I. Vaisman, *Locally conformal symplectic manifolds*, Internat. J. Math. Math. Sci. 8 (1985) 521-536.

[15] V. Vuletescu, *Blowing-up points of locally conformal Kähler manifolds*, Bull. Math. Soc. Sci. Math. Roumanie. 52 (2009) 387-390.

[16] X. Yang, G. Zhao, *A note on the Morse-Novikov cohomology of blow-ups of locally conformal Kähler manifolds*, Bull. Aust. Math. Soc. 91 (2015) 155-166.

CENTER FOR APPLIED MATHEMATICS, TIANJIN UNIVERSITY, TIANJIN 300072, P. R. CHINA

E-mail address: syang.math@gmail.com, syangmath@tju.edu.cn

DEPARTMENT OF MATHEMATICS, CHONGQING UNIVERSITY, CHONGQING 401331, P. R. CHINA

E-mail address: xiangdongyang2009@gmail.com, math.yang@cqu.edu.cn

DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU 610064, P. R. CHINA

E-mail address: gszhao@scu.edu.cn