Remarks on the Thickness of $K_{n,n,n}$

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Abstract The thickness $\theta(G)$ of a graph $G$ is the minimum number of planar subgraphs into which $G$ can be decomposed. In this paper, we provide a new upper bound for the thickness of the complete tripartite graphs $K_{n,n,n}$ ($n \geq 3$) and obtain $\theta(K_{n,n,n}) = \left\lceil \frac{n+1}{3} \right\rceil$, when $n \equiv 3 \pmod{6}$.

Keywords thickness; complete tripartite graph; planar subgraphs decomposition.

Mathematics Subject Classification 05C10.

1 Introduction

The thickness $\theta(G)$ of a graph $G$ is the minimum number of planar subgraphs into which $G$ can be decomposed. It was defined by Tutte [10] in 1963, derived from early work on biplanar graphs [2,11]. It is a classical topological invariant of a graph and also has many applications to VLSI design, graph drawing, etc. Determining the thickness of a graph is NP-hard [6], so the results about thickness are few. The only types of graphs whose thicknesses have been determined are complete graphs [1,3], complete bipartite graphs [4] and hypercubes [5]. The reader is referred to [7,8] for more background on the thickness problems.

In this paper, we study the thickness of complete tripartite graphs $K_{n,n,n}$, ($n \geq 3$). When $n = 1, 2$, it is easy to see that $K_{1,1,1}$ and $K_{2,2,2}$ are planar graphs, so the thickness of both ones is one. Poranen proved $\theta(K_{n,n,n}) \leq \left\lceil \frac{n}{2} \right\rceil$ in [9] which was the only result about the thickness of $K_{n,n,n}$, as far as the author knows. We will give a new upper bound for $\theta(K_{n,n,n})$ and provide the exact number for the thickness of $K_{n,n,n}$, when $n$ is congruent to 3 mod 6, the main results of this paper are the following theorems.

Theorem 1. For $n \geq 3$, $\theta(K_{n,n,n}) \leq \left\lceil \frac{n+1}{3} \right\rceil + 1$.

Theorem 2. $\theta(K_{n,n,n}) = \left\lceil \frac{n+1}{3} \right\rceil$ when $n \equiv 3 \pmod{6}$.

2 The proofs of the theorems

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In [4], Beineke, Harary and Moon determined the thickness of complete bipartite graph $K_{m,n}$ for almost all values of $m$ and $n$.

**Lemma 3.**[4] The thickness of $K_{m,n}$ is $\left\lfloor \frac{mn}{2(m+n-2)} \right\rfloor$ except possibly when $m$ and $n$ are odd, $m \leq n$ and there exists an integer $k$ satisfying $n = \left\lfloor \frac{2(k(m-2)}{m-2k} \right\rfloor$.

**Lemma 4.** For $n \geq 3$, $\theta(K_{n,n,n}) \geq \left\lceil \frac{n+1}{3} \right\rceil$.

**Proof.** Since $K_{n,2n}$ is a subgraph of $K_{n,n,n}$, we have $\theta(K_{n,n,n}) \geq \theta(K_{n,2n})$. From Lemma 3, the thickness of $K_{n,2n}$ ($n \geq 3$) is $\left\lceil \frac{n+1}{3} \right\rceil$, so the lemma follows. \qed

For the complete tripartite graph $K_{n,n,n}$ with the vertex partition $(A,B,C)$, where $A = \{a_0, \ldots, a_{n-1}\}$, $B = \{b_0, \ldots, b_{n-1}\}$ and $C = \{c_0, \ldots, c_{n-1}\}$, we define a type of graphs, they are planar spanning subgraphs of $K_{n,n,n}$, denoted by $G[a_ib_j+c_k+i]$, in which $0 \leq i, j, k \leq n-1$ and all subscripts are taken modulo $n$. The graph $G[a_ib_j+c_k+i]$ consists of $n$ triangles $a_ib_j+c_k+i$ for $0 \leq i \leq n-1$ and six paths of length $n-1$, they are

$$a_0b_{j+1}c_{k+2}a_3b_{j+4}c_{k+5} \ldots a_3b_{j+3i+1}c_{k+3i+2} \ldots,$$

$$c_ka_{j+2}b_{j+3}a_4b_{j+5} \ldots c_{k+3}a_{3i+1}b_{j+3i+2} \ldots,$$

$$b_jc_{j+1}a_{b+3}b_{j+4}c_{k+3}a_{b+5} \ldots b_{j+3i}c_{k+3i+1}a_{3i+2} \ldots,$$

$$a_0c_{k+1}b_{j+2}a_3c_{k+4}b_{j+5} \ldots a_{3i}c_{k+3i+1}b_{j+3i+2} \ldots,$$

$$b_ja_{k+2}b_{j+3}a_4c_{k+5} \ldots b_{j+3i}a_{3i+1}c_{k+3i+2} \ldots,$$

$$c_kb_{j+1}a_2c_{k+3}b_{j+4}a_{5} \ldots c_{k+3}b_{j+3i+1}a_{3i+2} \ldots.$$

Equivalently, the graph $G[a_ib_j+c_k+i]$ is the graph with the same vertex set as $K_{n,n,n}$ and edge set

$$\{a_ib_{j+i-1}, a_ib_{j+i}, a_ib_{j+i+1}, a_ib_{k+i-1}, a_ib_{k+i}, a_ib_{k+i+1} \mid 1 \leq i \leq n-2\}$$

$$\cup\{b_{j+i}c_{k+i-1}, b_{j+i}c_{k+i}, b_{j+i}c_{k+i+1} \mid 1 \leq i \leq n-2\}$$

$$\cup\{a_0b_j, a_0b_{j+1}, a_{n-1}b_{j+n-2}, a_{n-1}b_{j+n-1}\}$$

$$\cup\{a_0c_k, a_0c_{k+1}, a_{n-1}c_{k+n-2}, a_{n-1}c_{k+n-1}\}$$

$$\cup\{b_jc_k, b_jc_{k+1}, b_{j+n-1}c_{k+n-2}, b_{j+n-1}c_{k+n-1}\}.$$ 

Figure 1(a) illustrates the planar spanning subgraph $G[a_ibjc]$ of $K_{5,5,5}$. 
Theorem 5. When \( n = 3p + 2 \) (\( p \) is a positive integer), \( \theta(K_{n,n,n}) \leq p + 2 \).

Proof. When \( n = 3p + 2 \) (\( p \) is a positive integer), we will construct two different planar subgraphs decompositions of \( K_{n,n,n} \) according to \( p \) is odd or even, in which the number of planar subgraphs is \( p + 2 \) in both cases.

Case 1. \( p \) is odd. Let \( G_1, \ldots, G_p \) be \( p \) planar subgraphs of \( K_{n,n,n} \) where \( G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)}], \) for \( 1 \leq t \leq \frac{p+1}{2} \); and \( G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)+2}], \) for \( \frac{p+3}{2} \leq t \leq p \) and \( p \geq 3 \). From the structure of \( G[a_i b_{j+c_{k+1}}], \) we get that no two edges in \( G_1, \ldots, G_p \) are repeated. Because subscripts in \( G_t, 1 \leq t \leq p \) are taken modulo \( n \), \( \{3(t - 1) \text{ (mod } n) \mid 1 \leq t \leq p\} = \{0, 3, 6, \ldots, 3(p - 1)\}, \) \( \{6(t - 1) \text{ (mod } n) \mid 1 \leq t \leq \frac{p+1}{2}\} = \{0, 6, \ldots, 3(p - 1)\} \) and \( \{6(t - 1) + 2 \text{ (mod } n) \mid \frac{p+3}{2} \leq t \leq p\} = \{3, 9, \ldots, 3(p-2)\} \), the subscript sets of \( b \) and \( c \) in \( G_t, 1 \leq t \leq p \) are the same, i.e.,
\{i + 3(t - 1) \pmod n \mid 1 \leq t \leq p\}
= \{i + 6(t - 1) \pmod n \mid 1 \leq t \leq \frac{p+1}{2}\} \cup \{i + 6(t - 1) + 2 \pmod n \mid \frac{p+3}{2} \leq t \leq p\}.

Furthermore, if there exists \(t \in \{1, \ldots, p\}\) such that \(a_ib_j\) is an edge in \(G_t\), then \(a_ic_j\) is an edge in \(G_k\) for some \(k \in \{1, \ldots, p\}\). If the edge \(a_ib_j\) is not in any \(G_t\), then neither is the edge \(a_ic_j\) in any \(G_t\), for \(1 \leq t \leq p\).

From the construction of \(G_t\), the edges that belong to \(K_{n,n,n}\) but not to any \(G_t\), \(1 \leq t \leq p\), are

\begin{align*}
a_0b_3(t-1)-1, & \quad a_0c_3(t-1)-1, \quad 1 \leq t \leq p \tag{1} \\
a_{n-1}b_3(t-1), & \quad a_{n-1}c_3(t-1), \quad 1 \leq t \leq p \tag{2} \\
a_ib_{i-3}, & \quad a_ib_{i-2}, \quad 0 \leq i \leq n-1 \tag{3} \\
a_ic_{i-3}, & \quad a_ic_{i-2}, \quad 0 \leq i \leq n-1 \tag{4} \\
b_ib_{i+3(t-1)-1}, & \quad b_ic_{i+3(t-1)}, \quad 0 \leq i \leq n-1 \text{ and } t = \frac{p+3}{2} \tag{5} \\
b_3(t-1)c_6(t-1)-1, & \quad b_3(t-1)-1c_6(t-1), \quad 1 \leq t \leq \frac{p+1}{2} \tag{6} \\
b_3(t-1)c_6(t-1)+1, & \quad b_3(t-1)-1c_6(t-1)+2, \quad \frac{p+3}{2} \leq t \leq p \text{ and } p \geq 3 \tag{7}
\end{align*}

Let \(G_{p+1}\) be the graph whose edge set consists of the edges in (3) and (5), and \(G_{p+2}\) be the graph whose edge set consists of the edges in (1), (2), (4), (6) and (7). In the following, we will describe plane drawings of \(G_{p+1}\) and \(G_{p+2}\).

(a) A planar embedding of \(G_{p+1}\).

Place vertices \(b_0, b_1, \ldots, b_{n-1}\) on a circle, place vertices \(a_{i+3}\) and \(c_{i+\frac{n+1}{2}}\) in the middle of \(b_i\) and \(b_{i+1}\), join each of \(a_{i+3}\) and \(c_{i+\frac{n+1}{2}}\) to both \(b_i\) and \(b_{i+1}\), we get a planar embedding of \(G_{p+1}\). For example, when \(p = 1, n = 5\), Figure 1(b) shows the subgraph \(G_2\) of \(K_{5,5,5}\).

(b) A planar embedding of \(G_{p+2}\).

Firstly, we place vertices \(c_0, c_1, \ldots, c_{n-1}\) on a circle, join vertex \(a_{i+3}\) to \(c_i\) and \(c_{i+1}\), for \(0 \leq i \leq n-1\), so that we get a cycle of length \(2n\). Secondly, join vertex \(a_{n-1}\) to \(c_{3(t-1)}\) for \(1 \leq t \leq p\), with lines inside of the cycle. Let \(\ell_i\) be the line drawn inside the cycle joining \(a_{n-1}\) with \(c_{6(t-1)-1}\) if \(1 \leq t \leq \frac{p+1}{2}\) or with \(c_{6(t-1)+1}\) if \(\frac{p+3}{2} \leq t \leq p\) \((p \geq 3)\). For \(1 \leq t \leq p\), insert the vertex \(b_{3(t-1)}\) in the line \(\ell_i\). Thirdly, join vertex \(a_0\) to \(c_{3(t-1)-1}\) for \(1 \leq t \leq p\), with lines outside of the cycle. Let \(\ell'_i\) be the line drawn outside the cycle joining \(a_0\) with \(c_{6(t-1)}\) if \(1 \leq t \leq \frac{p+1}{2}\) or with \(c_{6(t-1)+2}\) if \(\frac{p+3}{2} \leq t \leq p\) \((p \geq 3)\). For \(1 \leq t \leq p\), insert the vertex \(b_{3(t-1)-1}\) in the line \(\ell'_i\). In this way, we can get a planar embedding of \(G_{p+2}\). For example, when \(p = 1, n = 5\), Figure 1(c) shows the subgraph \(G_3\) of \(K_{5,5,5}\).

Summarizing, when \(p\) is an odd positive integer and \(n = 3p+2\), we get a decomposition of \(K_{n,n,n}\) into \(p+2\) planar subgraphs \(G_1, \ldots, G_{p+2}\).
Case 2. \( p \) is even. Let \( G_1, \ldots, G_p \) be \( p \) planar subgraphs of \( K_{n,n,n} \) where
\[
G_t = G[a_i b_i + 3(t-1) c_i t + 3] \quad \text{for} \quad 1 \leq t \leq \frac{p}{2}; \text{ and } G_t = G[a_i b_i + 3(t-1) c_i t + 2] \quad \text{for} \quad \frac{p+2}{2} \leq t \leq p.
\]
With a similar argument to the proof of Case 1, we can get that the subscript sets of \( b \) and \( c \) in \( G_t \), \( 1 \leq t \leq p \) are the same, i.e.,
\[
\{ i + 3(t-1) \text{ (mod } n \text{)} \mid 1 \leq t \leq p \}
\]
\[
= \{ i + 6(t-1) + 3 \text{ (mod } n \text{)} \mid 1 \leq t \leq \frac{p}{2} \} \cup \{ i + 6(t-1) + 2 \text{ (mod } n \text{)} \mid \frac{p+2}{2} \leq t \leq p \}.
\]
From the construction of \( G_t \), \( G_{\frac{p}{2}} \) and \( G_{\frac{p+2}{2}} \) have \( n - 2 \) edges in common, they are
\[
b_{i+3(\frac{p+2}{2} - 1)} c_i t + 3(\frac{p+2}{2} - 1) + 1, \quad 1 \leq i \leq n - 1 \text{ and } i \neq n - 4,
\]
we can delete them in one of these two graphs to avoid repetition.

The edges that belong to \( K_{n,n,n} \) but not to any \( G_t \), \( 1 \leq t \leq p \), are
\[
a_0 b_{0(t-1)-1}, \quad a_0 c_{0(t-1)-1}, \quad 1 \leq t \leq p \tag{8}
\]
\[
a^{-1} b_{3(t-1)}, \quad a^{-1} c_{3(t-1)}, \quad 1 \leq t \leq p \tag{9}
\]
\[
a_i b_{i-3}, \quad a_i b_{i-2}, \quad 0 \leq i \leq n - 1 \tag{10}
\]
\[
a_i c_{i-3}, \quad a_i c_{i-2}, \quad 0 \leq i \leq n - 1 \tag{11}
\]
\[
b_i c_{i-1}, \quad b_i c_i, \quad b_i c_{i+1}, \quad 0 \leq i \leq n - 1 \tag{12}
\]
\[
b_{3(t-1)} c_{6t-4}, \quad 1 \leq t \leq \frac{p}{2} \tag{13}
\]
\[
b_{3(t-1)} c_{6t-5}, \quad \frac{p+2}{2} \leq t \leq p \tag{14}
\]
\[
b_{3(t-1) - 1} c_{6t-3}, \quad 1 \leq t < \frac{p}{2} \tag{15}
\]
\[
b_{3(t-1) - 1} c_{6t-4}, \quad \frac{p+2}{2} \leq t \leq p \tag{16}
\]

Let \( G_{p+1} \) be the graph whose edge set consists of the edges in (10), (11) and (12), and \( G_{p+2} \) be the graph whose edge set consists of the edges in (8), (9), (13), (14), (15) and (16).
We draw \( G_{p+1} \) in the following way. Firstly, place vertices \( b_0, c_0, b_1, c_1, \ldots, b_{n-1}, c_{n-1} \) on a circle \( C \), join vertex \( c_i \) to \( b_i \) and \( b_{i+1} \), we get a cycle of length \( 2n \). Secondly, place vertices \( a_0, a_2, \ldots, a_{n-2} \) on a circle \( C' \) in the unbounded region defined by the circle \( C \) such that \( C \) is contained in the closed disk defined by \( C' \), place vertices \( a_1, a_3, \ldots, a_{n-1} \) on a circle \( C'' \) contained in the bounded region of \( C \). Join \( a_i \) to \( b_i - 3, b_i - 2, c_i - 3, \) and \( c_i - 2 \), join \( b_i \) to \( c_{i+1} \). We can get a planar embedding of \( G_{p+1} \), so it is a planar graph. \( G_{p+2} \) is also planar because it is a subgraph of a graph homeomorphic to a dipole (two vertices joined by some edges). For example, when \( p = 2, n = 8 \), Figure 2(c) and Figure 2(d) show the subgraphs \( G_3 \) and \( G_4 \) of \( K_8, 8, 8 \) respectively.

Summarizing, when \( p \) is an even positive integer and \( n = 3p + 2 \), we obtain a decomposition of \( K_{n,n,n} \) into \( p + 2 \) planar subgraphs \( G_1, \ldots, G_{p+2} \).
Theorem follows from Cases 1 and 2.

From the proof of Theorem 5, we draw planar subgraphs decompositions of $K_{5,5,5}$ and $K_{8,8,8}$ as illustrated in Figure 1 and Figure 2 respectively.

(a) The subgraph $G_1 = G[a_ib_ic_{i+3}]$ of $K_{8,8,8}$

(b) The subgraph $G_2 = b_4c_0 - b_5c_1 - b_6c_2 - b_0c_4 - b_1c_5 - b_2c_6$ of $K_{8,8,8}$ in which $G_2 = G[a_ib_{i+3}c_i]$
Proof of Theorem 1. Because \(K_{n-1,n-1,n-1}\) is a subgraph of \(K_{n,n,n}\), by Theorem 5, \(\theta(K_{n,n,n}) \leq p + 2\) also holds, when \(n = 3p\) or \(n = 3p + 1\) (\(p\) is a positive integer), the theorem follows.

Proof of Theorem 2. When \(n = 3p\) is odd, i.e., \(n \equiv 3 \mod 6\), we decompose \(K_{n,n,n}\) into \(p + 1\) planar subgraphs \(G_1, \ldots, G_{p+1}\), where \(G_t = G[a_ib_{i+3(t-1)c_{i+6(t-1)}}, \ldots, G_{p+1} = G[a_{n-1}b_{3(t-1)}c_{6(t-1)}], \ldots, G_{p+1} = G[a_{n-1}b_{3(t-1)}c_{6(t-1)}], \ldots, G_{p+1} = G[a_{n-1}b_{3(t-1)}c_{6(t-1)}]\), for \(1 \leq t \leq p\). With a similar argument to the proof of Theorem 5, we can get that the subscript sets of \(b\) and \(c\) in \(G_t\), \(1 \leq t \leq p\), are the same, i.e.,

\[
\{i + 3(t - 1) \mod n \mid 1 \leq t \leq p\} = \{i + 6(t - 1) \mod n \mid 1 \leq t \leq p\}.
\]

If the edge \(a_ib_j\) is in \(G_t\) for some \(t \in \{1, \ldots, p\}\), then there exists \(k \in \{1, \ldots, p\}\) such that \(a_ic_j\) is in \(G_k\). If the edge \(a_ib_j\) is not in any \(G_t\), then neither is the edge \(a_ic_j\) in any \(G_t\), for \(1 \leq t \leq p\).

From the construction of \(G_t = G[a_ib_{i+3(t-1)c_{i+6(t-1)}}, \ldots, G_{p+1} = G[a_{n-1}b_{3(t-1)}c_{6(t-1)}], \ldots, G_{p+1} = G[a_{n-1}b_{3(t-1)}c_{6(t-1)}]\), we list the edges that belong to \(K_{n,n,n}\) but not to any \(G_t\), \(1 \leq t \leq p\), as follows,

\[
a_0b_{3(t-1)-1}, \quad a_0c_{6(t-1)-1}, \quad 1 \leq t \leq p \tag{17}
\]

\[
a_{n-1}b_{3(t-1)}, \quad a_{n-1}c_{6(t-1)}, \quad 1 \leq t \leq p \tag{18}
\]

\[
b_{3(t-1)}c_{6(t-1)-1}, \quad b_{3(t-1)-1}c_{6(t-1)-1}, \quad 1 \leq t \leq p \tag{19}
\]

Let \(G_{p+1}\) be the graph whose edge set consists of the edges in (17), (18) and (19). It is easy to see that \(G_{p+1}\) is homeomorphic to a dipole and it is a planar graph.

Summarizing, when \(p\) is an odd positive integer and \(n = 3p\), we obtain a decomposition of \(K_{n,n,n}\) into \(p+1\) planar subgraphs \(G_1, \ldots, G_{p+1}\), therefor \(\theta(K_{n,n,n}) \leq p+1\). Combining this fact and Lemma 4, the theorem follows.
According to the proof of Theorem 2, we draw a planar subgraphs decomposition of $K_{3,3,3}$ as shown in Figure 3.

(a) The subgraph $G_1 = G[a_i,b_j,c_k]$ of $K_{3,3,3}$
(b) The subgraph $G_2$ of $K_{3,3,3}$

**Figure 3** A planar subgraphs decomposition of $K_{3,3,3}$

For some other $\theta(K_{n,n,n})$ with small $n$, combining Lemma 4 and Poranen’s result mentioned in Section 1, we have $\theta(K_{4,4,4}) = 2, \theta(K_{6,6,6}) = 3$. Since there exists a decomposition of $K_{7,7,7}$ with three planar subgraphs as shown in Figure 4, Lemma 4 implies that $\theta(K_{7,7,7}) = 3$. We also conjecture that the thickness of $K_{n,n,n}$ is $\lceil \frac{n+1}{3} \rceil$ for all $n \geq 3$.

(a) The subgraph $G_1 = G[a_i,b_j,c_k]$ of $K_{7,7,7}$
(b) The subgraph $G_2 - a_1b_2 - a_2b_3 - a_3b_4 - a_4b_5 - a_5b_6 - b_0c_1 - b_1c_2 - b_3c_4 - b_4c_5 - b_5c_6$

of $K_{7,7,7}$ in which $G_2 = G[a_i b_{i+2} c_{i+4}]$

(c) The subgraph $G_3$ of $K_{7,7,7}$

Figure 4  A planar subgraphs decomposition of $K_{7,7,7}$
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