# Remarks on the Thickness of $K_{n,n,n}$*

**Yan Yang**

Department of Mathematics
Tianjin University, Tianjin 300072, P.R.China
yanyang@tju.edu.cn

**Abstract** The thickness $\theta(G)$ of a graph $G$ is the minimum number of planar subgraphs into which $G$ can be decomposed. In this paper, we provide a new upper bound for the thickness of the complete tripartite graphs $K_{n,n,n}$ $(n \geq 3)$ and obtain $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$, when $n \equiv 3 \pmod 6$.

**Keywords** thickness; complete tripartite graph; planar subgraphs decomposition.

**Mathematics Subject Classification** 05C10.

## 1 Introduction

The *thickness* $\theta(G)$ of a graph $G$ is the minimum number of planar subgraphs into which $G$ can be decomposed. It was defined by Tutte [10] in 1963, derived from early work on biplanar graphs [2,11]. It is a classical topological invariant of a graph and also has many applications to VLSI design, graph drawing, etc. Determining the thickness of a graph is NP-hard [6], so the results about thickness are few. The only types of graphs whose thicknesses have been determined are complete graphs [1,3], complete bipartite graphs [4] and hypercubes [5]. The reader is referred to [7,8] for more background on the thickness problems.

In this paper, we study the thickness of complete tripartite graphs $K_{n,n,n}$, $(n \geq 3)$. When $n = 1, 2$, it is easy to see that $K_{1,1,1}$ and $K_{2,2,2}$ are planar graphs, so the thickness of both ones is one. Poranen proved $\theta(K_{n,n,n}) \leq \lceil \frac{n}{2} \rceil$ in [9] which was the only result about the thickness of $K_{n,n,n}$, as far as the author knows. We will give a new upper bound for $\theta(K_{n,n,n})$ and provide the exact number for the thickness of $K_{n,n,n}$, when $n$ is congruent to 3 mod 6, the main results of this paper are the following theorems.

**Theorem 1.** *For $n \geq 3$, $\theta(K_{n,n,n}) \leq \lceil \frac{n+1}{3} \rceil + 1$.*

**Theorem 2.** $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$ *when $n \equiv 3$ (mod 6).*

## 2 The proofs of the theorems

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In [4], Beineke, Harary and Moon determined the thickness of complete bipartite graph $K_{m,n}$ for almost all values of $m$ and $n$.

**Lemma 3.**[4] *The thickness of $K_{m,n}$ is $\lceil \frac{mn}{2(m+n-2)} \rceil$ except possibly when $m$ and $n$ are odd, $m \leq n$ and there exists an integer $k$ satisfying $n = \lfloor \frac{2k(m-2)}{m-2k} \rfloor$.*

**Lemma 4.** *For $n \geq 3$, $\theta(K_{n,n,n}) \geq \lceil \frac{n+1}{3} \rceil$.*

*Proof.* Since $K_{n,2n}$ is a subgraph of $K_{n,n,n}$, we have $\theta(K_{n,n,n}) \geq \theta(K_{n,2n})$. From Lemma 3, the thickness of $K_{n,2n}$ ($n \geq 3$) is $\lceil \frac{n+1}{3} \rceil$, so the lemma follows. □

For the complete tripartite graph $K_{n,n,n}$ with the vertex partition $(A, B, C)$, where $A = \{a_0, \ldots, a_{n-1}\}$, $B = \{b_0, \ldots, b_{n-1}\}$ and $C = \{c_0, \ldots, c_{n-1}\}$, we define a type of graphs, they are planar spanning subgraphs of $K_{n,n,n}$, denoted by $G[a_i b_{j+i} c_{k+i}]$, in which $0 \leq i, j, k \leq n-1$ and all subscripts are taken modulo $n$. The graph $G[a_i b_{j+i} c_{k+i}]$ consists of $n$ triangles $a_i b_{j+i} c_{k+i}$ for $0 \leq i \leq n-1$ and six paths of length $n-1$, they are

$$a_0 b_{j+1} c_{k+2} a_3 b_{j+4} c_{k+5} \ldots a_{3i} b_{j+3i+1} c_{k+3i+2} \cdots,$$

$$c_k a_1 b_{j+2} c_{k+3} a_4 b_{j+5} \ldots c_{k+3i} a_{3i+1} b_{j+3i+2} \cdots,$$

$$b_j c_{k+1} a_2 b_{j+3} c_{k+4} a_5 \ldots b_{j+3i} c_{k+3i+1} a_{3i+2} \cdots,$$

$$a_0 c_{k+1} b_{j+2} a_3 c_{k+4} b_{j+5} \ldots a_{3i} c_{k+3i+1} b_{j+3i+2} \cdots,$$

$$b_j a_1 c_{k+2} b_{j+3} a_4 c_{k+5} \ldots b_{j+3i} a_{3i+1} c_{k+3i+2} \cdots,$$

$$c_k b_{j+1} a_2 c_{k+3} b_{j+4} a_5 \ldots c_{k+3i} b_{j+3i+1} a_{3i+2} \cdots.$$

Equivalently, the graph $G[a_i b_{j+i} c_{k+i}]$ is the graph with the same vertex set as $K_{n,n,n}$ and edge set

$$\{a_i b_{j+i-1}, a_i b_{j+i}, a_i b_{j+i+1}, a_i c_{k+i-1}, a_i c_{k+i}, a_i c_{k+i+1} \mid 1 \leq i \leq n-2\}$$

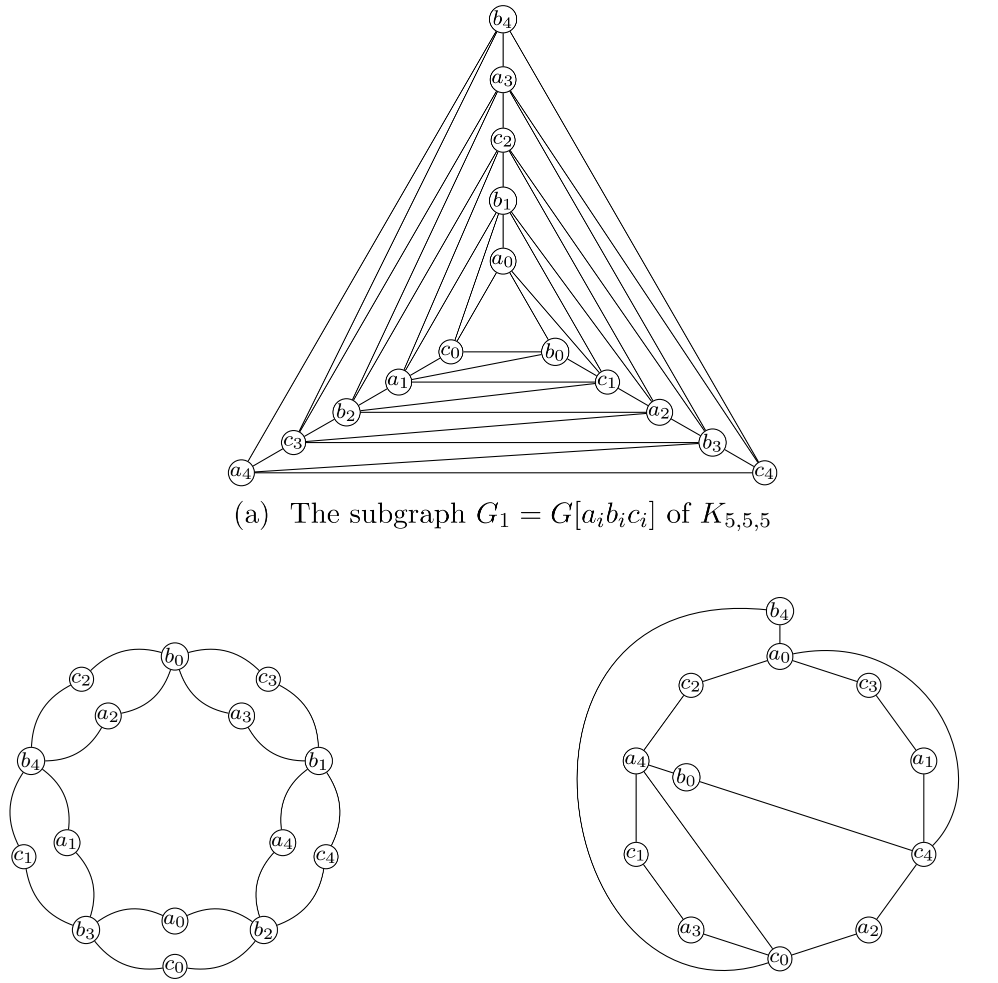
$$\cup \{b_{j+i} c_{k+i-1}, b_{j+i} c_{k+i}, b_{j+i} c_{k+i+1} \mid 1 \leq i \leq n-2\}$$

$$\cup \{a_0 b_j, a_0 b_{j+1}, a_{n-1} b_{j+n-2}, a_{n-1} b_{j+n-1}\}$$

$$\cup \{a_0 c_k, a_0 c_{k+1}, a_{n-1} c_{k+n-2}, a_{n-1} c_{k+n-1)}\}$$

$$\cup \{b_j c_k, b_j c_{k+1}, b_{j+n-1} c_{k+n-2}, b_{j+n-1} c_{k+n-1}\}.$$

Figure 1(a) illustrates the planar spanning subgraph $G[a_i b_i c_i]$ of $K_{5,5,5}$.




(a) The subgraph $G_1 = G[a_ib_ic_i]$ of $K_{5,5,5}$

(b) The subgraph $G_2$ of $K_{5,5,5}$

(c) The subgraph $G_3$ of $K_{5,5,5}$

**Figure 1** A planar subgraphs decomposition of $K_{5,5,5}$

**Theorem 5.** *When* $n = 3p + 2$ *(*$p$ *is a positive integer),* $\theta(K_{n,n,n}) \leq p + 2$.

*Proof.* When $n = 3p+2$ ($p$ is a positive integer), we will construct two different planar subgraphs decompositions of $K_{n,n,n}$ according to $p$ is odd or even, in which the number of planar subgraphs is $p + 2$ in both cases.

**Case 1.** $p$ is odd. Let $G_1, \ldots, G_p$ be $p$ planar subgraphs of $K_{n,n,n}$ where $G_t = G[a_ib_{i+3(t-1)}c_{i+6(t-1)}]$, for $1 \leq t \leq \frac{p+1}{2}$; and $G_t = G[a_ib_{i+3(t-1)}c_{i+6(t-1)+2}]$, for $\frac{p+3}{2} \leq t \leq p$ and $p \geq 3$. From the structure of $G[a_ib_{j+i}c_{k+i}]$, we get that no two edges in $G_1, \ldots, G_p$ are repeated. Because subscripts in $G_t, 1 \leq t \leq p$ are taken modulo $n$, $\{3(t-1) \pmod{n} \mid 1 \leq t \leq p\} = \{0, 3, 6, \ldots, 3(p-1)\}$, $\{6(t-1) \pmod{n} \mid 1 \leq t \leq \frac{p+1}{2}\} = \{0, 6, \ldots, 3(p-1)\}$ and $\{6(t-1)+2 \pmod{n} \mid \frac{p+3}{2} \leq t \leq p\} = \{3, 9, \ldots, 3(p-2)\}$, the subscript sets of $b$ and $c$ in $G_t$, $1 \leq t \leq p$ are the same, i.e.,

$$\{i+3(t-1) \text{ (mod n)} \mid 1 \le t \le p\}$$
$$= \{i+6(t-1) \text{ (mod n)} \mid 1 \le t \le \frac{p+1}{2}\} \cup \{i+6(t-1)+2 \text{ (mod n)} \mid \frac{p+3}{2} \le t \le p\}.$$

Furthermore, if there exists $t \in \{1, \ldots, p\}$ such that $a_ib_j$ is an edge in $G_t$, then $a_ic_j$ is an edge in $G_k$ for some $k \in \{1, \ldots, p\}$. If the edge $a_ib_j$ is not in any $G_t$, then neither is the edge $a_ic_j$ in any $G_t$, for $1 \le t \le p$.

From the construction of $G_t$, the edges that belong to $K_{n,n,n}$ but not to any $G_t$, $1 \le t \le p$, are

$$a_0b_{3(t-1)-1}, \quad a_0c_{3(t-1)-1}, \qquad 1 \le t \le p \tag{1}$$

$$a_{n-1}b_{3(t-1)}, \quad a_{n-1}c_{3(t-1)}, \qquad 1 \le t \le p \tag{2}$$

$$a_ib_{i-3}, \quad a_ib_{i-2}, \qquad 0 \le i \le n-1 \tag{3}$$

$$a_ic_{i-3}, \quad a_ic_{i-2}, \qquad 0 \le i \le n-1 \tag{4}$$

$$b_ic_{i+3(t-1)-1}, \quad b_ic_{i+3(t-1)}, \qquad 0 \le i \le n-1 \text{ and } t = \frac{p+3}{2} \tag{5}$$

$$b_{3(t-1)}c_{6(t-1)-1}, \quad b_{3(t-1)-1}c_{6(t-1)}, \qquad 1 \le t \le \frac{p+1}{2} \tag{6}$$

$$b_{3(t-1)}c_{6(t-1)+1}, \quad b_{3(t-1)-1}c_{6(t-1)+2}, \qquad \frac{p+3}{2} \le t \le p \text{ and } p \ge 3 \tag{7}$$

Let $G_{p+1}$ be the graph whose edge set consists of the edges in (3) and (5), and $G_{p+2}$ be the graph whose edge set consists of the edges in $(1), (2), (4), (6)$ and $(7)$. In the following, we will describe plane drawings of $G_{p+1}$ and $G_{p+2}$.

**(a)** A planar embedding of $G_{p+1}$.
Place vertices $b_0, b_1, \ldots, b_{n-1}$ on a circle, place vertices $a_{i+3}$ and $c_{i+\frac{n+1}{2}}$ in the middle of $b_i$ and $b_{i+1}$, join each of $a_{i+3}$ and $c_{i+\frac{n+1}{2}}$ to both $b_i$ and $b_{i+1}$, we get a planar embedding of $G_{p+1}$. For example, when $p = 1$, $n = 5$, Figure 1(b) shows the subgraph $G_2$ of $K_{5,5,5}$.
**(b)** A planar embedding of $G_{p+2}$.
Firstly, we place vertices $c_0, c_1, \ldots, c_{n-1}$ on a circle, join vertex $a_{i+3}$ to $c_i$ and $c_{i+1}$, for $0 \le i \le n-1$ , so that we get a cycle of length $2n$. Secondly, join vertex $a_{n-1}$ to $c_{3(t-1)}$ for $1 \le t \le p$, with lines inside of the cycle. Let $\ell_t$ be the line drawn inside the cycle joining $a_{n-1}$ with $c_{6(t-1)-1}$ if $1 \le t \le \frac{p+1}{2}$ or with $c_{6(t-1)+1}$ if $\frac{p+3}{2} \le t \le p$ $(p \ge 3)$. For $1 \le t \le p$, insert the vertex $b_{3(t-1)}$ in the line $\ell_t$. Thirdly, join vertex $a_0$ to $c_{3(t-1)-1}$ for $1 \le t \le p$, with lines outside of the cycle. Let $\ell'_t$ be the line drawn outside the cycle joining $a_0$ with $c_{6(t-1)}$ if $1 \le t \le \frac{p+1}{2}$ or with $c_{6(t-1)+2}$ if $\frac{p+3}{2} \le t \le p$ $(p \ge 3)$. For $1 \le t \le p$, insert the vertex $b_{3(t-1)-1}$ in the line $\ell'_t$. In this way, we can get a planar embedding of $G_{p+2}$. For example, when $p = 1$, $n = 5$, Figure 1(c) shows the subgraph $G_3$ of $K_{5,5,5}$.

Summarizing, when $p$ is an odd positive integer and $n = 3p+2$, we get a decomposition of $K_{n,n,n}$ into $p+2$ planar subgraphs $G_1, \ldots, G_{p+2}$.

**Case 2.** $p$ is even. Let $G_1, \ldots, G_p$ be $p$ planar subgraphs of $K_{n,n,n}$ where $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)+3}]$, for $1 \leq t \leq \frac{p}{2}$; and $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)+2}]$, for $\frac{p+2}{2} \leq t \leq p$. With a similar argument to the proof of Case 1, we can get that the subscript sets of $b$ and $c$ in $G_t$, $1 \leq t \leq p$ are the same, i.e.,

$$\{i + 3(t-1) \text{ (mod n)} \mid 1 \leq t \leq p\}$$

$$= \{i + 6(t-1) + 3 \text{ (mod n)} \mid 1 \leq t \leq \frac{p}{2}\} \cup \{i + 6(t-1) + 2 \text{ (mod n)} \mid \frac{p+2}{2} \leq t \leq p\}.$$

From the construction of $G_t$, $G_{\frac{p}{2}}$ and $G_{\frac{p+2}{2}}$ have $n-2$ edges in common, they are $b_{i+3(\frac{p+2}{2}-1)} c_{i+6(\frac{p+2}{2}-1)+1}$, $1 \leq i \leq n-1$ and $i \neq n-4$, we can delete them in one of these two graphs to avoid repetition.

The edges that belong to $K_{n,n,n}$ but not to any $G_t$, $1 \leq t \leq p$, are

$$a_0 b_{3(t-1)-1}, \quad a_0 c_{3(t-1)-1}, \qquad 1 \leq t \leq p \tag{8}$$

$$a_{n-1} b_{3(t-1)}, \quad a_{n-1} c_{3(t-1)}, \qquad 1 \leq t \leq p \tag{9}$$

$$a_i b_{i-3}, \quad a_i b_{i-2}, \qquad 0 \leq i \leq n-1 \tag{10}$$

$$a_i c_{i-3}, \quad a_i c_{i-2}, \qquad 0 \leq i \leq n-1 \tag{11}$$

$$b_i c_{i-1}, \quad b_i c_i, \quad b_i c_{i+1}, \qquad 0 \leq i \leq n-1 \tag{12}$$

$$b_{3(t-1)} c_{6t-4}, \qquad 1 \leq t \leq \frac{p}{2} \tag{13}$$

$$b_{3(t-1)} c_{6t-5}, \qquad \frac{p+2}{2} < t \leq p \tag{14}$$

$$b_{3(t-1)-1} c_{6t-3}, \qquad 1 \leq t < \frac{p}{2} \tag{15}$$

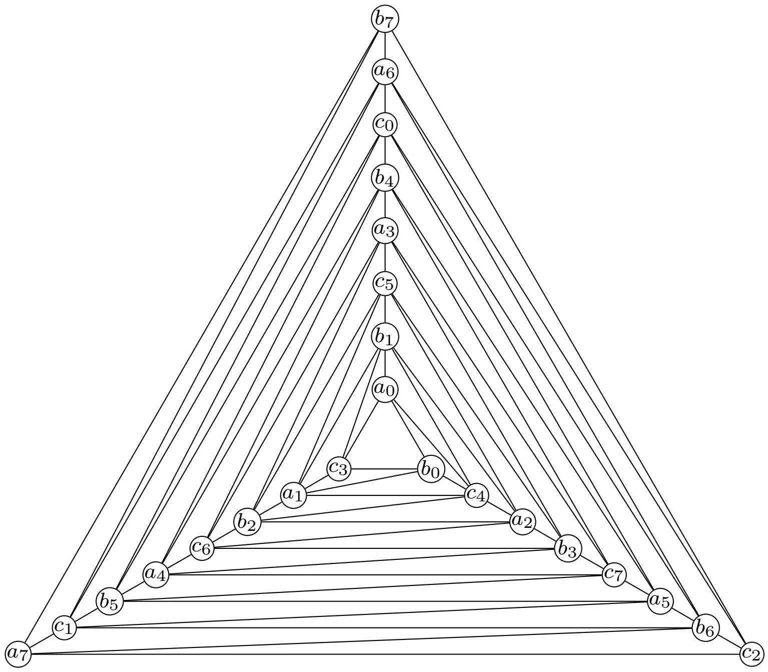
$$b_{3(t-1)-1} c_{6t-4}, \qquad \frac{p+2}{2} \leq t \leq p \tag{16}$$

Let $G_{p+1}$ be the graph whose edge set consists of the edges in (10), (11) and (12), and $G_{p+2}$ be the graph whose edge set consists of the edges in (8), (9), (13), (14), (15) and (16). We draw $G_{p+1}$ in the following way. Firstly, place vertices $b_0, c_0, b_1, c_1, \ldots, b_{n-1}, c_{n-1}$ on a circle $C$, join vertex $c_i$ to $b_i$ and $b_{i+1}$, we get a cycle of length $2n$. Secondly, place vertices $a_0, a_2, \ldots, a_{n-2}$ on a circle $C'$ in the unbounded region defined by the circle $C$ such that $C$ is contained in the closed disk defined by $C'$, place vertices $a_1, a_3, \ldots, a_{n-1}$ on a circle $C''$ contained in the bounded region of $C$. Join $a_i$ to $b_{i-3}, b_{i-2}, c_{i-3}$, and $c_{i-2}$, join $b_i$ to $c_{i+1}$. We can get a planar embedding of $G_{p+1}$, so it is a planar graph. $G_{p+2}$ is also planar because it is a subgraph of a graph homeomorphic to a dipole (two vertices joined by some edges). For example, when $p = 2$, $n = 8$, Figure 2(c) and Figure 2(d) show the subgraphs $G_3$ and $G_4$ of $K_{8,8,8}$ respectively.

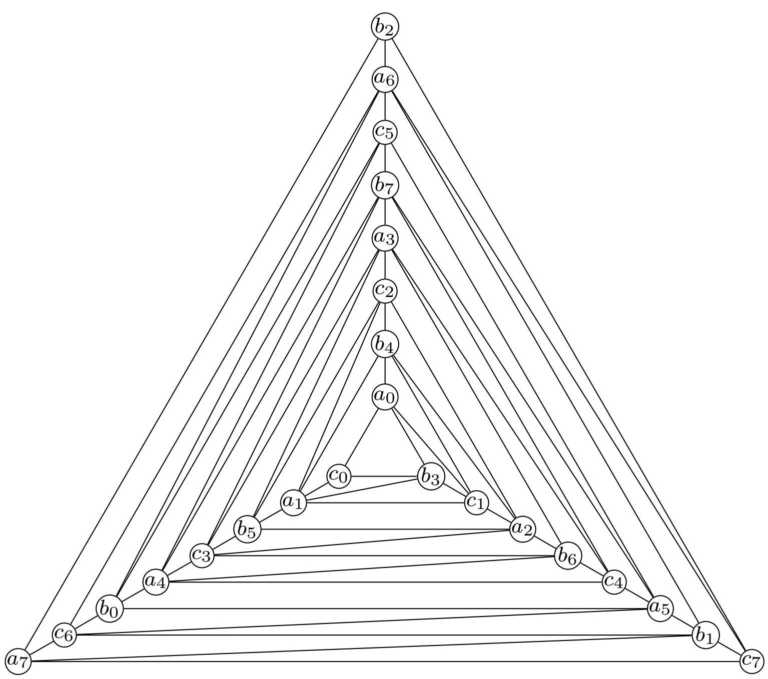
Summarizing, when $p$ is an even positive integer and $n = 3p + 2$, we obtain a decomposition of $K_{n,n,n}$ into $p + 2$ planar subgraphs $G_1, \ldots, G_{p+2}$.

Theorem follows from Cases 1 and 2. □

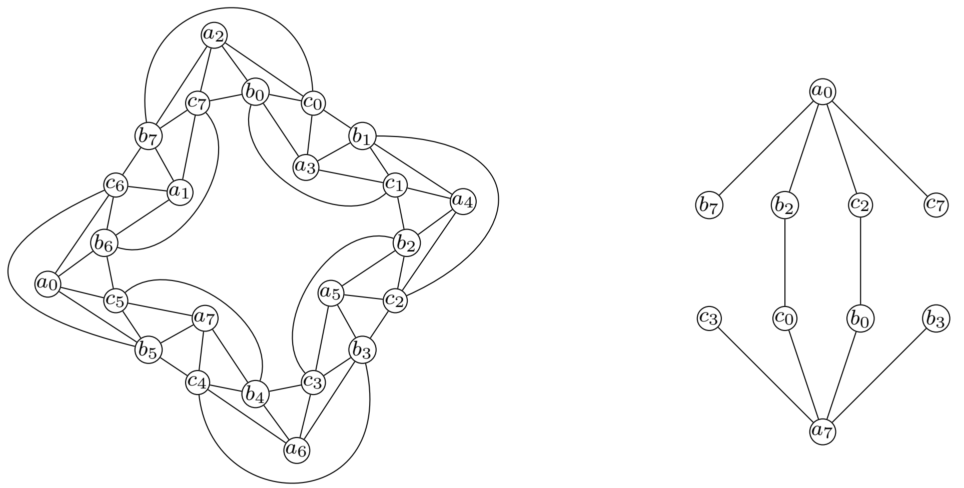
From the proof of Theorem 5, we draw planar subgraphs decompositions of $K_{5,5,5}$ and $K_{8,8,8}$ as illustrated in Figure 1 and Figure 2 respectively.



(a) The subgraph $G_1 = G[a_i b_i c_{i+3}]$ of $K_{8,8,8}$



(b) The subgraph $G_2 - b_4c_0 - b_5c_1 - b_6c_2 - b_0c_4 - b_1c_5 - b_2c_6$ of $K_{8,8,8}$ in which $G_2 = G[a_i b_{i+3} c_i]$




(c) The subgraph $G_3$ of $K_{8,8,8}$ (d) The subgraph $G_4$ of $K_{8,8,8}$

**Figure 2** A planar subgraphs decomposition of $K_{8,8,8}$

**Proof of Theorem 1.** Because $K_{n-1,n-1,n-1}$ is a subgraph of $K_{n,n,n}$, $\theta(K_{n-1,n-1,n-1}) \leq \theta(K_{n,n,n})$, by Theorem 5, $\theta(K_{n,n,n}) \leq p+2$ also holds, when $n = 3p$ or $n = 3p+1$ ($p$ is a positive integer), the theorem follows. □

**Proof of Theorem 2.** When $n = 3p$ is odd, i.e., $n \equiv 3 \pmod 6$, we decompose $K_{n,n,n}$ into $p+1$ planar subgraphs $G_1, \ldots, G_{p+1}$, where $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)}]$, for $1 \leq t \leq p$. With a similar argument to the proof of Theorem 5, we can get that the subscript sets of $b$ and $c$ in $G_t$, $1 \leq t \leq p$ are the same, i.e.,

$$\{i + 3(t-1) \text{ (mod n)} \mid 1 \leq t \leq p\} = \{i + 6(t-1) \text{ (mod n)} \mid 1 \leq t \leq p\}.$$

If the edge $a_i b_j$ is in $G_t$ for some $t \in \{1, \ldots, p\}$, then there exists $k \in \{1, \ldots, p\}$ such that $a_i c_j$ is in $G_k$. If the edge $a_i b_j$ is not in any $G_t$, then neither is the edge $a_i c_j$ in any $G_t$, for $1 \leq t \leq p$.

From the construction of $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)}]$, we list the edges that belong to $K_{n,n,n}$ but not to any $G_t$, $1 \leq t \leq p$, as follows.

$$a_0 b_{3(t-1)-1}, \quad a_0 c_{6(t-1)-1}, \qquad 1 \leq t \leq p \tag{17}$$

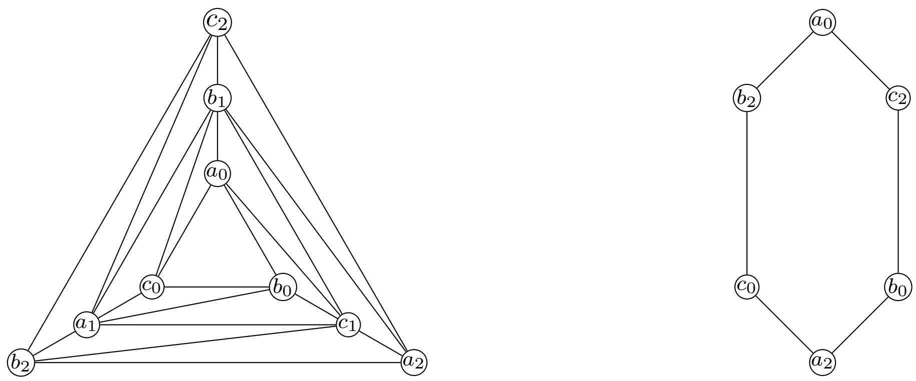
$$a_{n-1} b_{3(t-1)}, \quad a_{n-1} c_{6(t-1)}, \qquad 1 \leq t \leq p \tag{18}$$

$$b_{3(t-1)} c_{6(t-1)-1}, \quad b_{3(t-1)-1} c_{6(t-1)}, \qquad 1 \leq t \leq p \tag{19}$$

Let $G_{p+1}$ be the graph whose edge set consists of the edges in (17), (18) and (19). It is easy to see that $G_{p+1}$ is homeomorphic to a dipole and it is a planar graph.

Summarizing, when $p$ is an odd positive integer and $n = 3p$, we obtain a decomposition of $K_{n,n,n}$ into $p+1$ planar subgraphs $G_1, \ldots, G_{p+1}$, therefor $\theta(K_{n,n,n}) \leq p+1$. Combining this fact and Lemma 4, the theorem follows. □

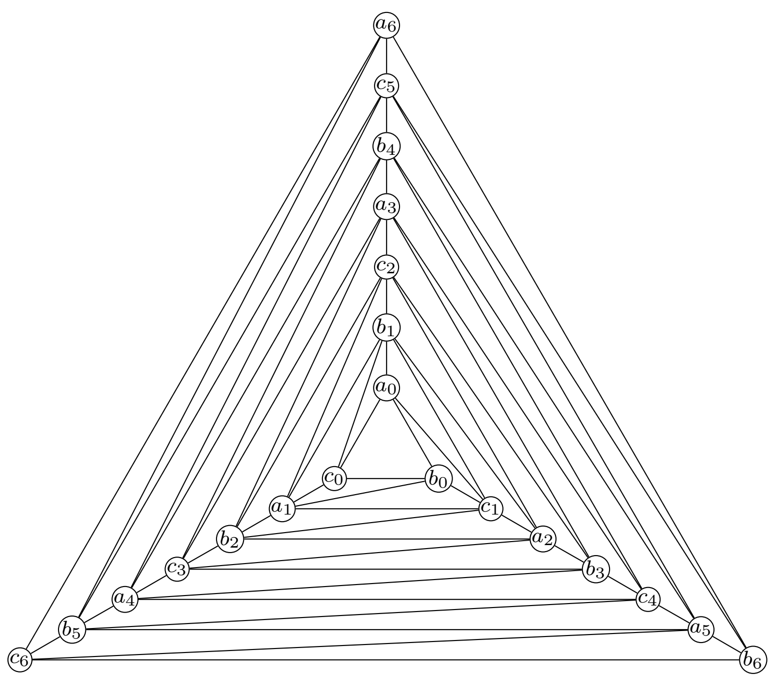
According to the proof of Theorem 2, we draw a planar subgraphs decomposition of $K_{3,3,3}$ as shown in Figure 3.



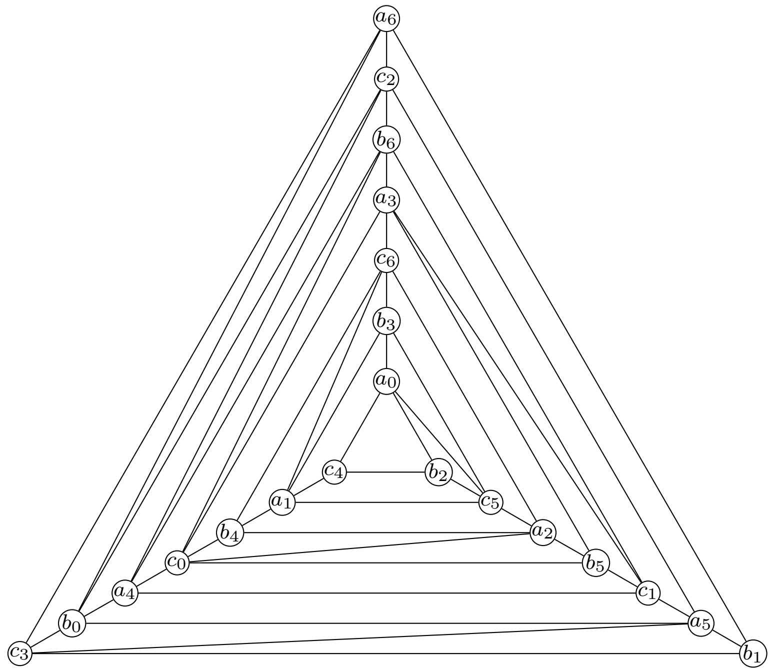

(a) The subgraph $G_1 = G[a_i b_i c_i]$ of $K_{3,3,3}$ (b) The subgraph $G_2$ of $K_{3,3,3}$

**Figure 3** A planar subgraphs decomposition of $K_{3,3,3}$

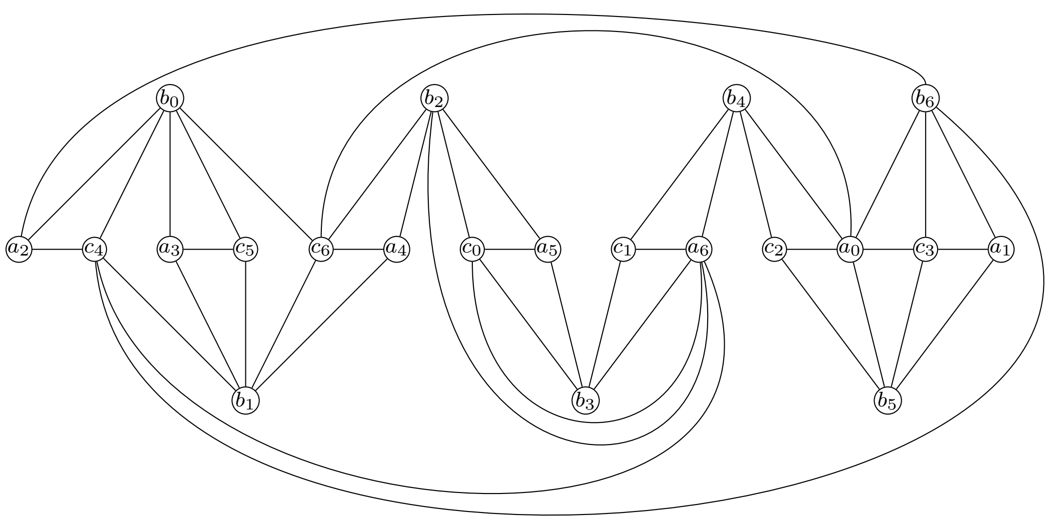
For some other $\theta(K_{n,n,n})$ with small $n$, combining Lemma 4 and Poranen's result mentioned in Section 1, we have $\theta(K_{4,4,4}) = 2, \theta(K_{6,6,6}) = 3$. Since there exists a decomposition of $K_{7,7,7}$ with three planar subgraphs as shown in Figure 4, Lemma 4 implies that $\theta(K_{7,7,7}) = 3$. We also conjecture that the thickness of $K_{n,n,n}$ is $\lceil \frac{n+1}{3} \rceil$ for all $n \geq 3$.




(a) The subgraph $G_1 = G[a_i b_i c_i]$ of $K_{7,7,7}$




(b) The subgraph $G_2 - a_1b_2 - a_2b_3 - a_3b_4 - a_4b_5 - a_5b_6 - b_0c_1 - b_1c_2 - b_3c_4 - b_4c_5 - b_5c_6$ of $K_{7,7,7}$ in which $G_2 = G[a_ib_{i+2}c_{i+4}]$




(c) The subgraph $G_3$ of $K_{7,7,7}$

**Figure 4** A planar subgraphs decomposition of $K_{7,7,7}$

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